Non-extensive Thermodynamics

T.S.Biró, G.G.Barnaföldi and P.Ván

Wigner Research Centre for Physics of the Hungarian Academy of Science,
Budapest, Hungary, EU

arXiv: 1208.2533v2

PACS: 12.38.Mh, 12.40.Ee, 05.70.-a

Talk at DM2012: November 29, 2012, Johannesburg, South-Africa.
Slides by TSB, last update: 2012.11.10. 17:04 Budapest, Hungary

Content

- **Motivation:** q-phenomenology or extended thermodynamics; temperature fluctuations
- **Derivation:** from improved canonical treatment to Rényi and Tsallis formula
- **Application:** the physics of the reservoir: QGP in MIT bag, black hole, chaotic YM fields
- **Conclusion:** Physical interpretation of the fitted spectral temperature and power

Ideal Gas

T.S.Biró

Wigner Research Centre for Physics of the Hungarian Academy of Science,
Budapest, Hungary, EU

PACS: 12.38.Mh, 12.40.Ee, 05.70.-a

Auxiliary material, October 29, 2012, Braga, Portugal.
Slides by TSB, last update: 2012.11.10. 17:05

Content

- **Fundation:** cut power-law single-particle energy distribution; Gamma temperature fluctuations
- **Derivation:** from constant heat capacity eos to canonical Rényi and Tsallis formula
- **Discussion:** positive, negative, infinite and zero heat-capicity
- **Conclusion:** Physical interpretation of the power-law fitted spectral temperature and power

Constant heat capacity eos

- $C = \frac{dE}{dT} = C_0,$
- $T = T_0 + \frac{1}{C_0}\, E$
- $S = \int \frac{dE}{T} = S_0 + C_0 \, \ln\left(1 + \frac{E}{C_0 T_0}\right)$
- $P = e^{-S(E)} = K_0 \left(1 + \frac{E}{C_0 T_0}\right)^{-C_0}$ **thermodynamical probability 1/W**

Constant heat capacity eos

Probability of stand-alone „microcanonical"

- $P(E_1) = e^{-S(E_1)} = K_0 \left(1 + \frac{E_1}{C_0 T_0}\right)^{-C_0}$

Conditional probability of a subsystem of it:

- $P(E_1|E - E_1) = \frac{P(E)}{P(E-E_1)} = e^{S(E-E_1)-S(E)} =$

$$\left(1 - \frac{E_1}{C_0 T_0 + E}\right)^{C_0}$$

Constant heat capacity eos

Probability of stand-alone „microcanonical”

- $P(E_1) = e^{-S(E_1)} = K_0 \left(1 + \frac{E_1}{C_0 T_0}\right)^{-C_0}$

Conditional probability of a subsystem of it:

- $P(E_1 | E - E_1) = \frac{P(E)}{P(E - E_1)} = e^{S(E - E_1) - S(E)} =$

$$\left(1 - \frac{E_1}{C_0 T_0 + E}\right)^{C_0}$$

$C_0 T$

Constant heat capacity eos

Two **different** subsystem-state probabilities:

- $$P(E_1) \neq P(E_1 | E - E_1)$$

Because entropy is not additive!

- $S(E) \neq S(E_1) + S(E - E_1)$

Constant heat capacity eos

Mutual information (*based on joint and marginal prob.-s*):

$$I_{12} = \sum_{i,j} r_{ij} \log \frac{r_{ij}}{p_i q_j} \;, \quad p_i = \sum_j r_{ij}, \quad q_j = \sum_i r_{ij}$$

Mutual information (*based on entropy*)

- $I_S(E_1|E) \; = S(E_1) + S(E - E_1) \; - S(E)$

Zero mutual information (*based on additive* **formal log**)

- $I_K(E_1|E) \; = K\big(S(E_1)\big) + K\big(S(E - E_1)\big) - K\big(S(E)\big) = 0$

Constant heat capacity eos

Factorizing „***deformed thermodynamical probability***”:

$$\boldsymbol{P_K(E) = e^{-K(S(E))}}$$

In this way (*the part of an ideal gas is an ideal gas…*)

- $\boldsymbol{P_K}(E_1) = \boldsymbol{P_K}(E_1 | E - E_1) = e^{-K(S(E_1))}$

From zero mutual information (*based on additive* **formal log**)

- $K\big(S(E_1)\big) = K\big(S(E)\big) - K\big(S(E - E_1)\big)$

Constant heat capacity eos

Factorizing „*deformed thermodynami*... *ity*":

$$P(E) = e^{-K(S(E))}$$

In thi...

Fr... *on additive* **formal log**)

- K ... $)) - K\big(S(E - E_1)\big)$

Call it „improved canonical"

Constant heat capacity eos

Formal logarithm, alias „*deformed entropy*”:

$$\boldsymbol{K(S(E)) = \frac{e^{aS}-1}{a} = \frac{E}{T_0}, \quad if \quad a = \frac{1}{C_0}}$$

In this way (*the part of an ideal gas is an ideal gas…*)

- $\boldsymbol{P_K}(E) = e^{-K(S(E))} = e^{-E/T_0}$

From the new entropy ⟹ *factorizing subsystem*

$$P_K(\mathrm{E}) = P_K(E_1) \cdot P_K(\mathrm{E} - E_1)$$

Constant heat capacity eos

Its Gibbs ensemble average is „**Tsallis-entropy**”:

$$K\big(S(E)\big) = \sum_i p_i K(-\ln p_i) \quad = \frac{1}{a}\sum_i \left(p_i^{1-a} - p_i \right)$$

The original S functional is „**Rényi-entropy**”:

$$S(E) = K^{-1}\Big(\sum_i p_i K(-\ln p_i)\Big) \quad = \frac{1}{a} \ln \sum_i p_i^{1-a}$$

Constant heat capacity eos

Formal logarithm, alias „*deformed entropy*" for conditional prob.:

$$K^*\big(S(E)\big) = \frac{1-e^{-aS}}{a} = \frac{E}{T}, \quad if \quad a = \frac{1}{C_0}$$

In this way (*the part of an ideal gas is an ideal gas...*)

- $P_{K*}(E) = e^{-K^*(S(E))} = e^{-E/T}$

From the new entropy ⇨ *factorizing subsystem*

$$P_{K*}(\mathrm{E}) = P_{K*}(E_1) \cdot P_{K*}(\mathrm{E} - E_1)$$

Constant heat capacity eos

Its Gibbs ensemble average is „**Tsallis-entropy**":

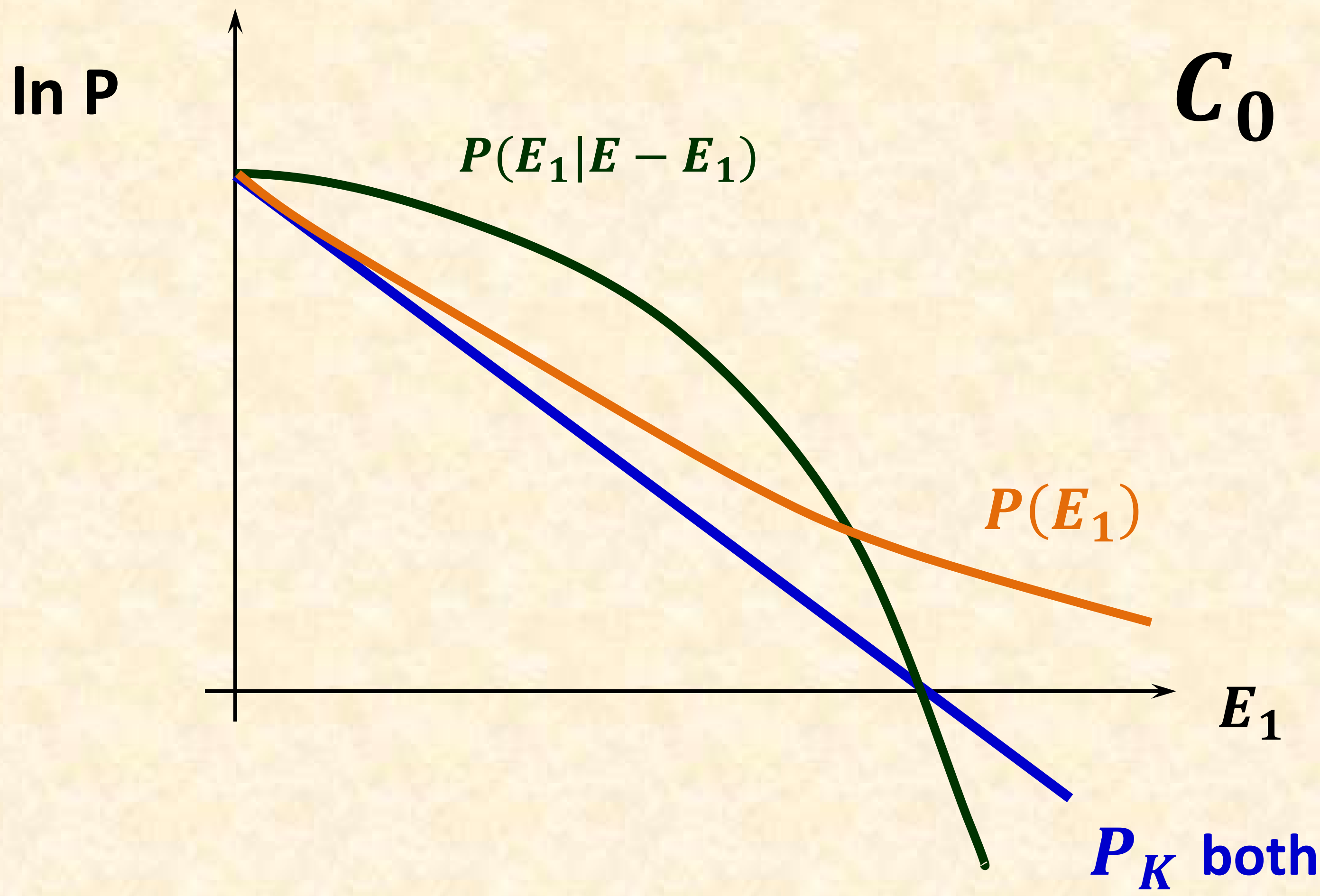
$$K^*\big(S(E)\big) = \sum_i p_i K^*(-\ln p_i) = \frac{1}{a}\sum_i (p_i - p_i^{1+a})$$

The original S functional is „**Rényi-entropy**":

$$S(E) = K^{*\,-1}\Big(\sum_i p_i K^*(-\ln p_i)\Big) = \frac{1}{a}\ln\sum_i p_i^{1+a}$$

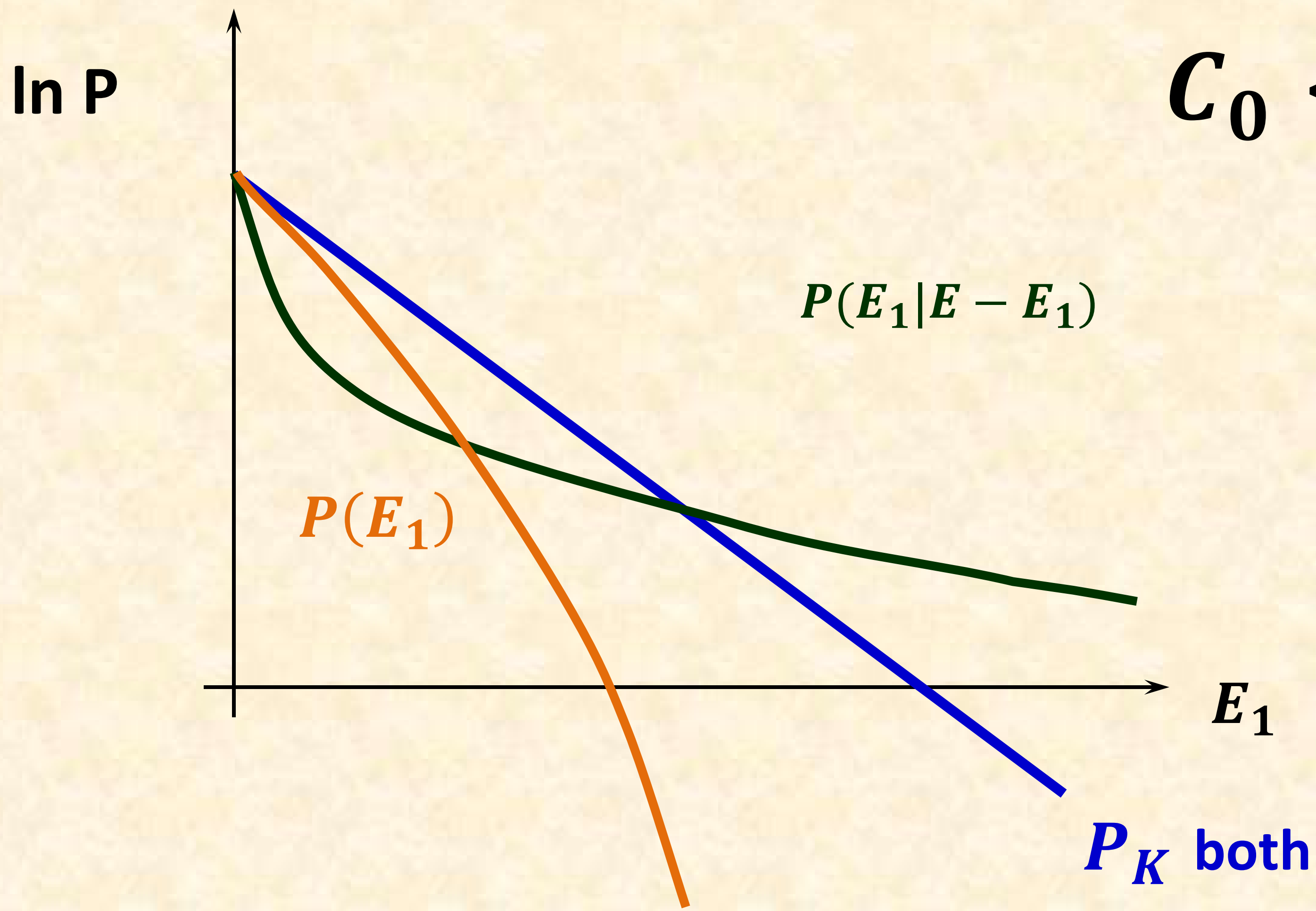
Ideal Gas: subsystem energy distributions (spectra)

$C_0 > 0$



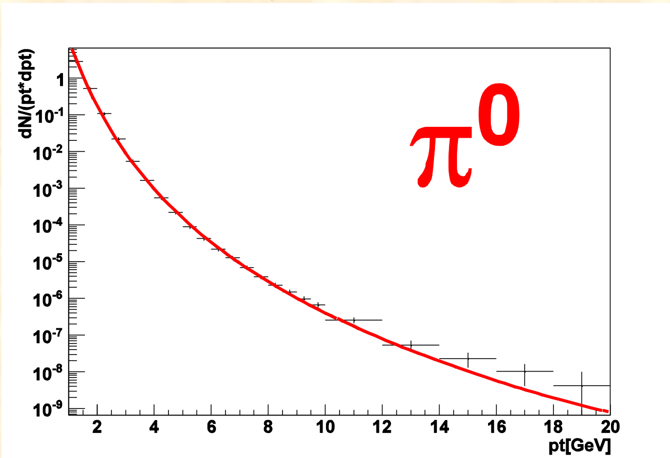
Ideal Gas: subsystem energy distributions (spectra)

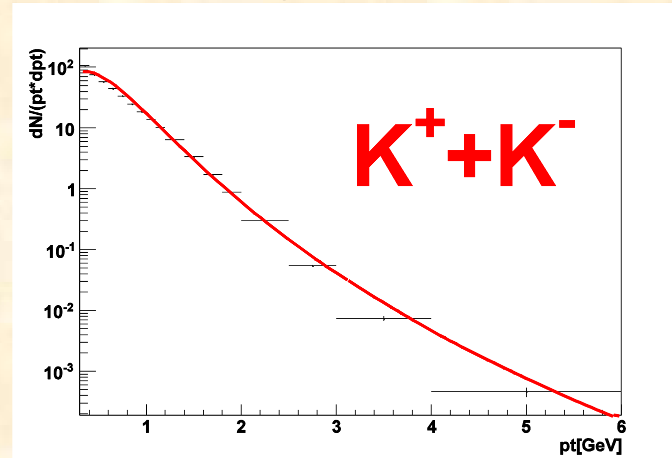
$C_0 < \ 0$

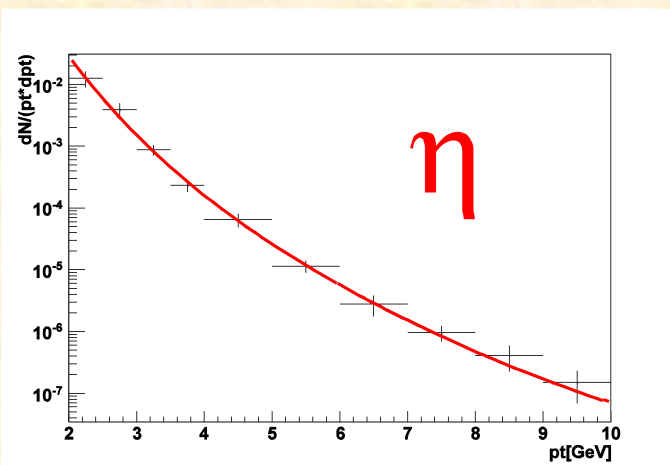


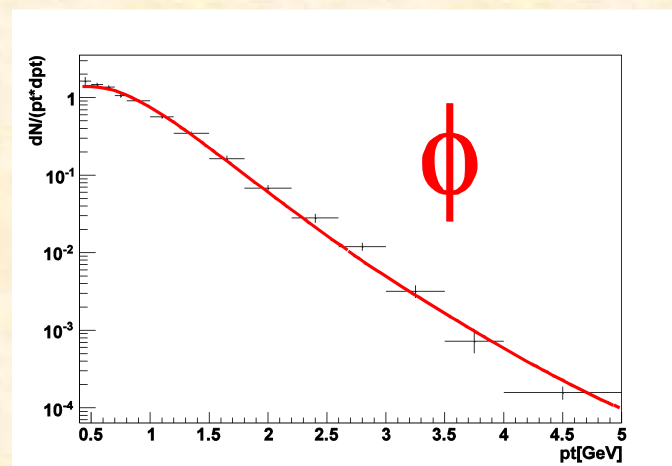
Experiment: RHIC Star and Phenix data

Theoretical model: Tsallis quark matter + transverse flow + quark coalescence fits to hadron spectra





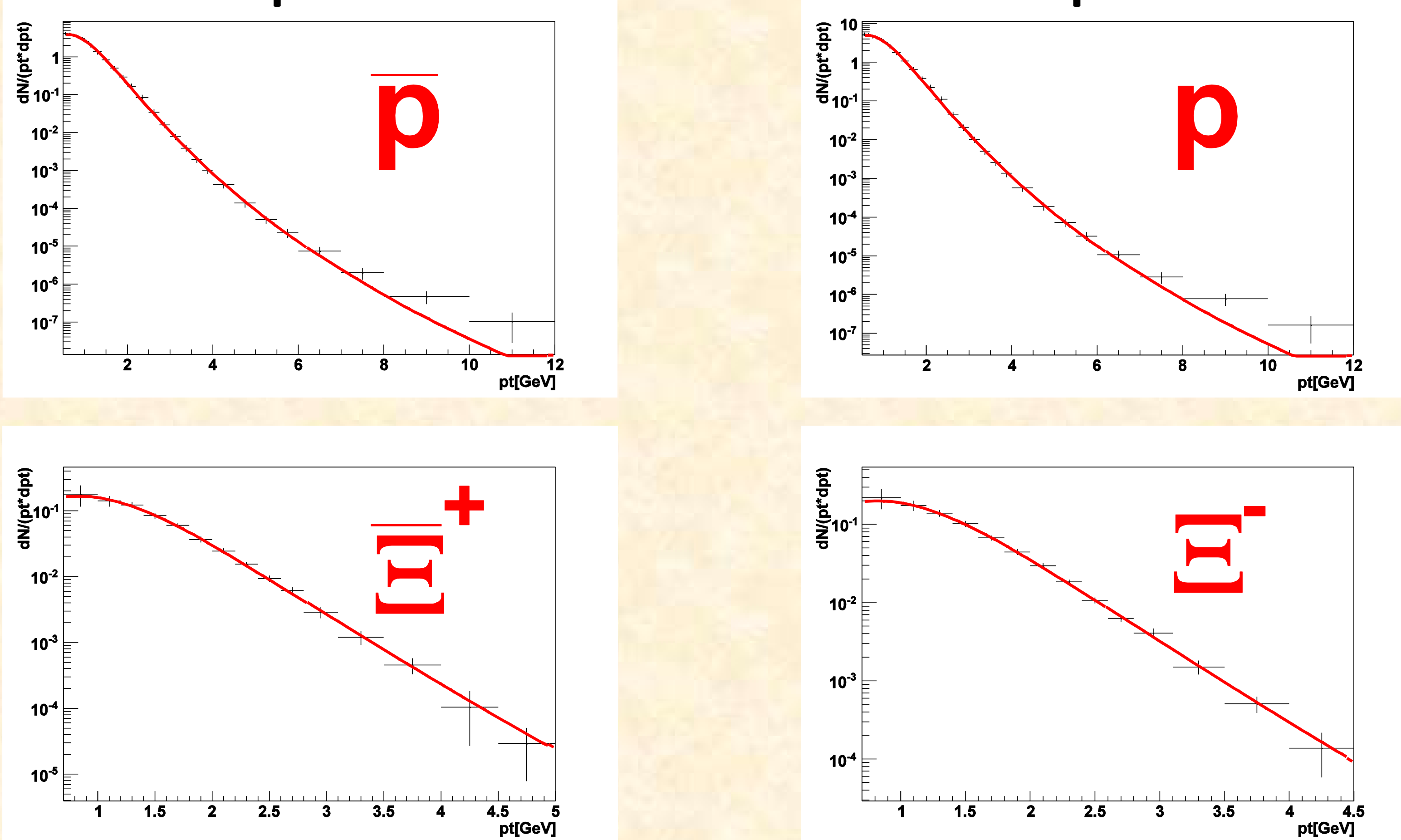




T.S.Biró, K.Ürmössy, JPhysG 36, 064044, 2009 (SQM 2008, Beijing)

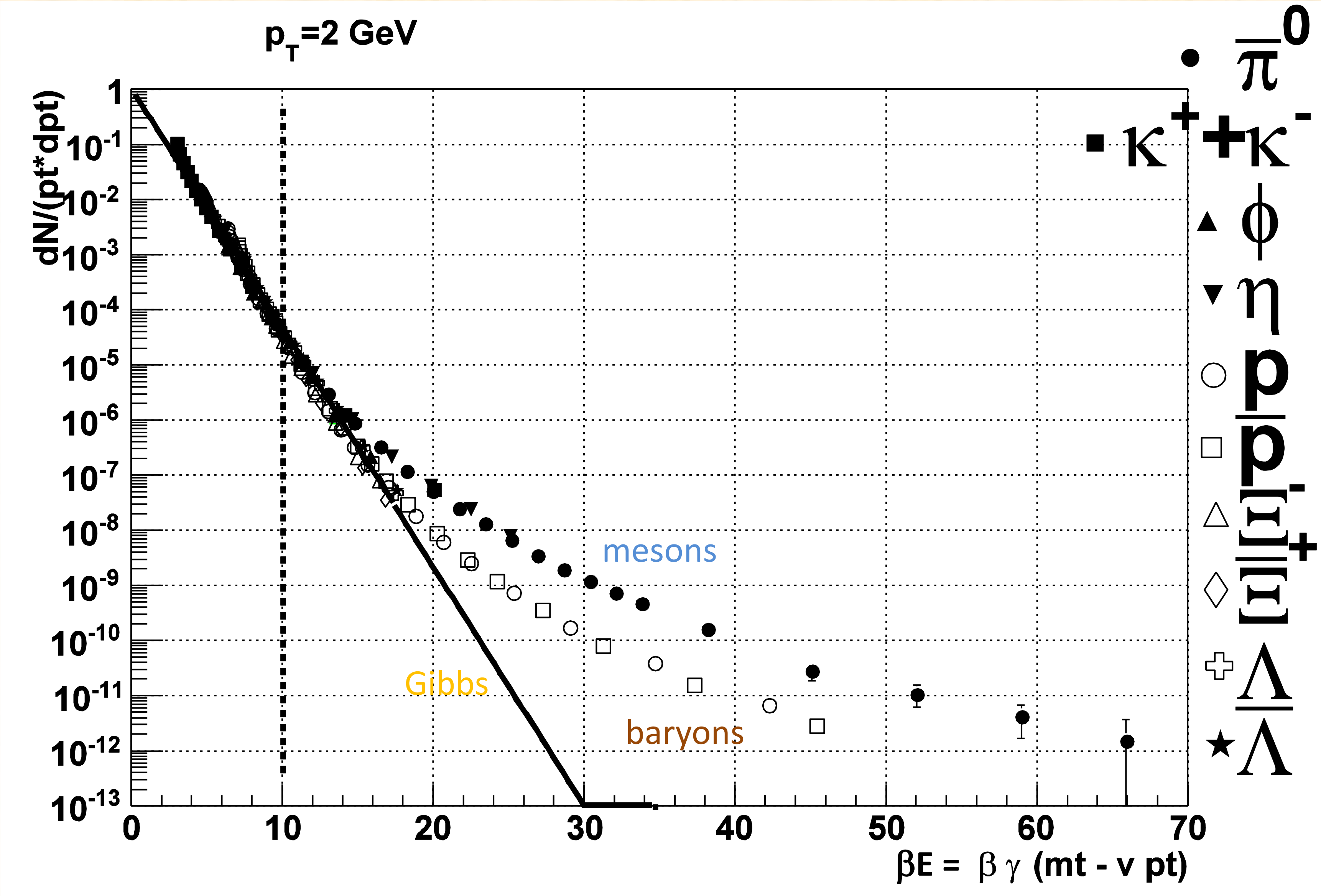
Experiment: RHIC Star and Phenix data

Theoretical model: Tsallis quark matter + transverse flow + quark coalescence fits to hadron spectra

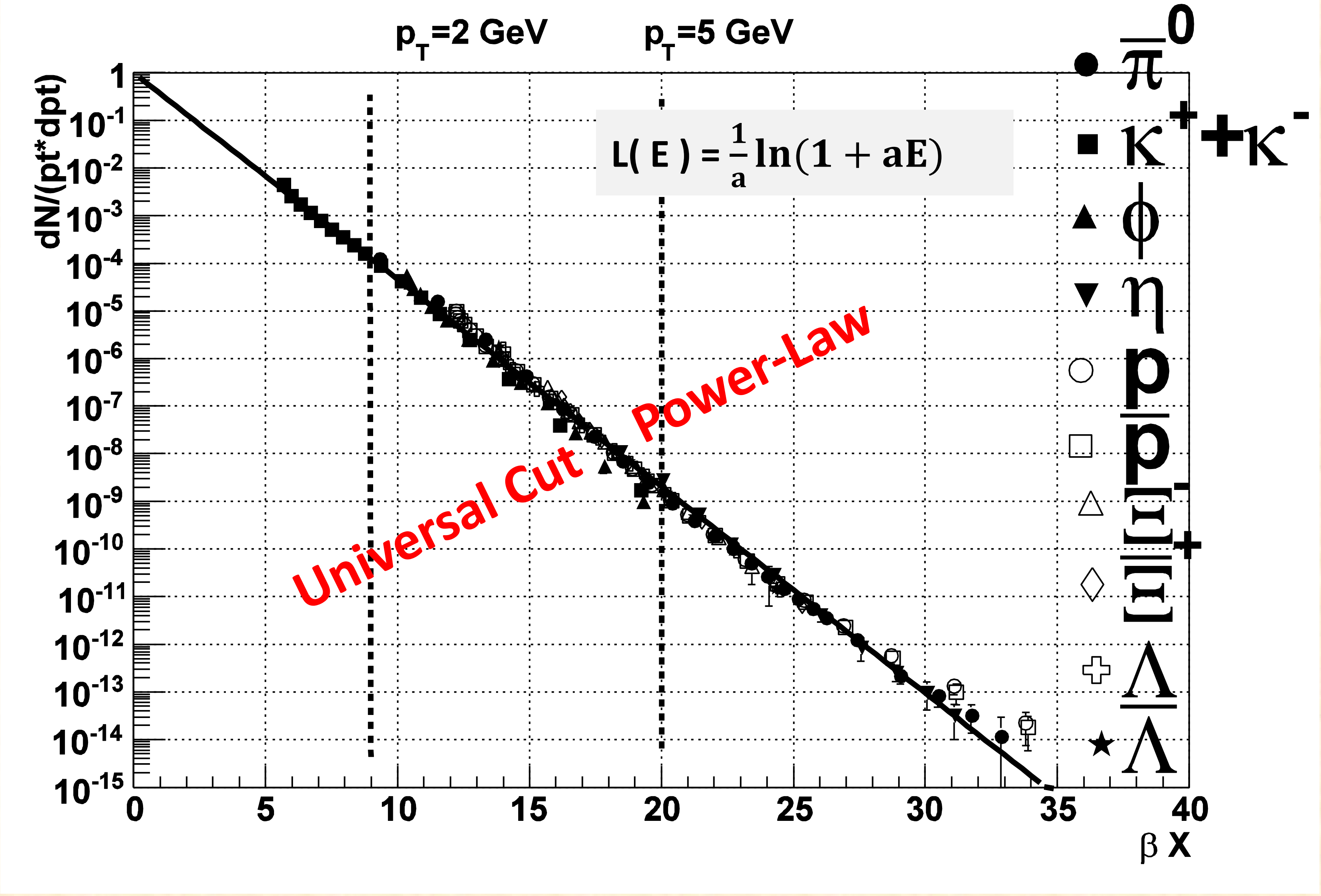


T.S.Biró, K.Ürmössy, JPhysG 36, 064044, 2009 (SQM 2008, Beijing)

Not an exponential in E_1 !



Exp of a Log is Power-Law



Quark coalescence: weighted convolution

Meson = two quarks;

$$\iint \left(1+\beta\frac{E_1}{C}\right)^{-C}\left(1+\beta\frac{E_2}{C}\right)^{-C}|\varphi(E_1-E_2)|^2\,\delta(E_1+E_2-E)$$

$$\approx \frac{1}{32}\,|\varphi(0)|^2\left(1+\beta\frac{E}{2C}\right)^{-2C}$$

T.S.Biró, K.Ürmössy, JPhysG 36, 064044, 2009 (SQM 2008, Beijing)

Quark coalescence: slope addition

Meson q-1 = quark (q-1)/2;
Baryon q-1 = quark (q-1)/3;

$$f^n\left(\frac{E}{n}\right) = \left(1 + \beta\,\frac{E}{nC}\right)^{-nC}$$

T.S.Biró, K.Ürmössy, JPhysG 36, 064044, 2009 (SQM 2008, Beijing)

Quark Coalescence

Meson = 2 quarks,   Baryon = 3 quarks

$$P_{meson}(E) \approx P_{quark}\left(\frac{E}{2}\right) \cdot P_{quark}\left(\frac{E}{2}\right)$$

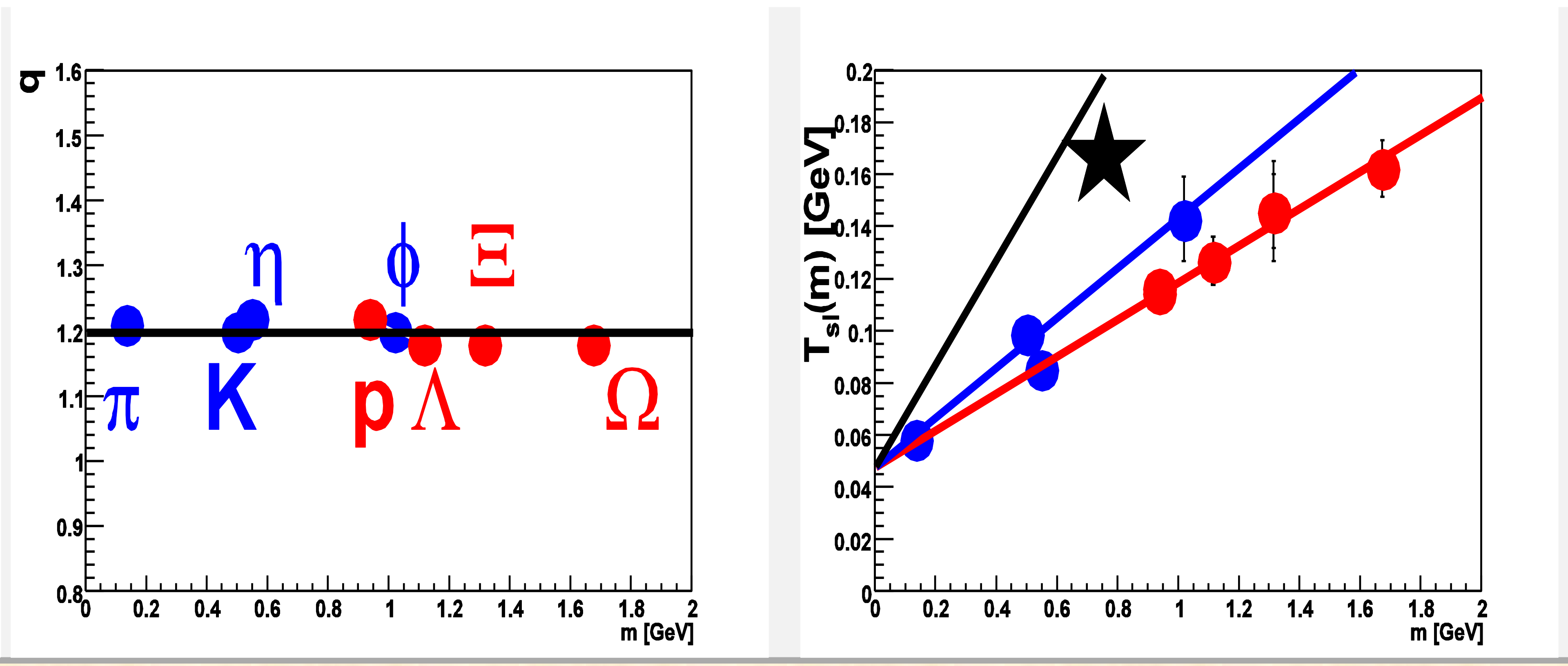
$$P_{baryon}(E) \approx P_{quark}\left(\frac{E}{3}\right) \cdot P_{quark}\left(\frac{E}{3}\right) \cdot P_{quark}\left(\frac{E}{3}\right)$$

Consequence:

$$\mathfrak{T}_{meson}(E) = \mathfrak{T}_{quark}\left(\frac{E}{2}\right) = T_0 + {}^{E}/_{2C},$$

$$\mathfrak{T}_{baryon}(E) = \mathfrak{T}_{quark}\left(\frac{E}{3}\right) = T_0 + {}^{E}/_{3C},$$

Rising slopes with energy and mass



Meson q-1 = quark (q-1)/2;
Baryon q-1 = quark (q-1)/3;
Mini BH q-1 = $2/\pi^2$

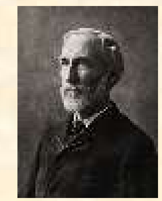
T.S.Biró, K.Ürmössy, JPhysG 36, 064044, 2009 (SQM 2008, Beijing)

Power-Law Tails and Entropy Formulas

- **Motivation:** phenomenology and theory
  - Canonical **Power-Law tailed Energy Spectra**
  - Fits **to AA, pp and e+e- (** EPJA 40, 299, 2009; PLB 701, 111, 2011 **)**
  - **Rényi- and Tsallis (and more)** Entropy Formulas, beta-fluctuations, **superstatistics**
  - **Thermodynamics constraints possible formulas via** allowed composition **rules (** PRE 83, 061147, 2011 **)**
  - **Scaled repetition leads to associative composition (**EPL 84, 56003, 2008 **)**
  - **2 body → maxprob → weighted sum of many copies:** Gibbs' **ensemble average formula (** arXiv 1208. 2533, 1209.5963 **)**

Entropy formulas

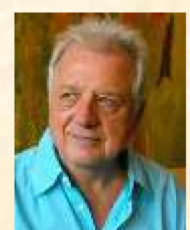
- $S = ln \frac{N!}{\prod_i N_i!}$ Boltzmann (permutation)



- $S = -\sum P_i \, ln \, P_i$ Gibbs (Planck)

- $S = \frac{1}{1-q} \ln \sum P_i^q$ Rényi
- $S = \frac{1}{q-1} \sum (P_i - P_i^q)$ Tsallis (Chravda, Aczél, Daróczy,...)

There are (much) more !



Canonical distribution with Rényi entropy

$$\frac{1}{1-q}\ln\sum p_i^q - \alpha\sum p_i - \beta\sum p_i E_i = \max$$

$$\frac{1}{1-q}\frac{q\,p_i^{q-1}}{\sum p_i^q} = \alpha + \beta E_i$$

This cut power-law distribution is

an **excellent** fit to particle spectra

in high-energy experiments!

$$p_i = \frac{1}{e^{\hat{L}(S)}}\left(1+(1-q)\frac{\beta(E_i - \langle E\rangle)}{q}\right)^{\frac{1}{q-1}}$$

Canonical distribution with Tsallis entropy

$$\frac{1}{1-q}\sum(p_i^q - p_i) - \alpha\sum p_i - \beta\sum p_i E_i = \max$$

$$\frac{1}{1-q}\, q\, p_i^{q-1} = \alpha + \frac{1}{1-q} + \beta E_i$$

$$p_i = \left(Z^{1-q} + (1-q)\frac{\beta E_i}{q} \right)^{\frac{1}{q-1}}$$

This cut power-law distribution is an excellent fit to particle spectra in high-energy experiments!

Superstatistics:

NBD = Euler ⊗ Poisson

Power Law = Euler ⊗ Gibbs

$$\binom{-k-1}{n}(-f)^n(1+f)^{-k-1-n}=\int_0^\infty \frac{(x\,f)^n}{n!}e^{-f\,x}\cdot\frac{x^k}{k!}e^{-x}\,dx$$

$$\left(1+\frac{\beta E}{k}\right)^{-k-1}=\int_0^\infty e^{-\frac{\beta E_i}{k}x}\cdot\frac{x^k}{k!}e^{-x}\,dx$$

$$q=\frac{k}{k+1}$$

Interpretation: eventy by event multiplicity fluctuations;
Volume fluctuations;
Temperature fluctuations;

Why to use the Tsallis / Rényi entropy formulas? ☺

- It generalizes the Boltzmann-Gibbs-Shannon formula
- It treats statistical entanglement between subsystem and reservoir (due to conservation)
- It claims to be universal (applicable for whatever material quality of the reservoir)
- It leads to a cut power-law energy distribution in the canonical treatment

Why not to use the Tsallis / Rényi entropy formulas? ☹

- They lack 300 years of classical thermodynamic foundation
- Tsallis is not additive, Rényi is not linear
- There is an extra parameter q (mysterious?)
- How do different q systems equilibrate ?
- Why this and not any other ?
- It looks pretty much formal…

Derivation as improved canonical

- **Derivation:**
  - **Microcanonical entropy maximum for two**
  - **Reservoir-independent temperature: the best one can (see also: Almeida, Physica A300, 424, 2001)**
  - **Which composition rule leads to higher order agreement *(cannot be the simple addition)***
  - **Make the choice of the additive L(S) universal → separation constant = 1 / heat capacity**
  - **Result: L(S) is Tsallis entropy, S is Rényi entropy**

General Derivation: formulas

- Two bodies: $\mathrm{K}\big(S(E_1)\big) + \ K\big(S(E - E_1)\big) \ = \ \ max.$
- Zeroth Law: $\beta_1 = K'(S(E_1)) \cdot S'(E_1)$

$$= K'\big(S(E \ - E_1)\big) \cdot S'(E - E_1)$$

Taylor: $S(E \ - E_1) = S(E) - E_1\, S'(E) + \frac{1}{2} E_1^2\, S''(E) \ldots$

Derivation: formulas

$$\beta_1 = K'(S(E)) \cdot S'(E)$$
$$- \; E_1 \left[S'(E)^2 K''\big(S(E)\big) + S''(E) K'(S(E))\right]$$

The content of the bracket be **zero**!

Derivation: formulas

$$\beta = K'(S(E)) \cdot S'(E)$$

and the content of the bracket [ ] is **zero**:

$$\frac{K''(S)}{K'(S)} = -\frac{S''(E)}{S'(E)^2} = \frac{1}{C(E)}$$

Universal if constant:

$$\boldsymbol{\frac{K''(S)}{K'(S)} = a}$$

Derivation: formulas

The solution is:

$$K(S) = \frac{e^{aS} - 1}{a}$$

This generates

$$\boldsymbol{K(-ln\, P_i) = \frac{1}{a}\,(P_i^{-a} - 1)}$$

Derivation: formulas

- Generalized to an ensemble of subsystems:

$$S = -\sum_i P_i \ln P_i \quad \rightarrow \quad K(S) = \sum_i P_i \, K(-\ln P_i)$$

Derivation: Tsallis entropy

The canonical principle becomes:

$$\frac{1}{a}\sum\left(P_i^{1-a} - P_i\right) - \beta \sum P_i\, E_i - \alpha \sum P_i = max.$$

The entropy with q = 1-a

$$K(S) = S_{Tsallis} = \frac{1}{q-1}\sum\left(P_i - P_i^q\right)$$

Derivation: Rényi entropy

The Rényi entropy is the original one,

but the Tsallis entropy is to be maximized canonically

$$S_{Rényi} = K^{-1}(\, S_{Tsallis} \,) \; = \; \frac{1}{1-q}\, ln \sum P_i^q$$

Improved Canonical Distribution

- $P_i = \left(Z^{1-q} + (1-q)\frac{\beta}{q}\, E_i\right)^{\frac{1}{q-1}}$
- Expressed by the reservoir's physical parameters via using our results:
- $P_i = \frac{1}{Z}\left(1 + \frac{Z^{-1/C}}{C-1}\, e^{S/C}\, \frac{1}{T} E_i\right)^{-C}$

Check infinite C limit!

Improved Canonical Distribution

- $P_i = \left(Z^{1-q} + (1-q)\frac{\beta}{q}\, E_i\right)^{\frac{1}{q-1}}$
- Expressed by the reservoir's physical parameters via using our results:
- $P_i = \frac{1}{Z}\left(1 + \frac{Z^{-1/C}}{C-1} e^{S/C}\, \frac{1}{T} E_i\right)^{-C}$

β

Check infinite C limit!

Improved Canonical Distribution

- Slope of the energy distribution:
- $\frac{1}{\mathfrak{T}} = - \frac{d}{dE_i} \ln P_i, \qquad \mathfrak{T} = T_0 + \frac{E_i}{C}$
- Expressed by the reservoir's physical parameters via using our results:
- $T_0 = T\, e^{-S/C}\, Z^{1/C}\, (1 - {}^{1}/_{C})$
- $ln_q\, Z = C\left(Z^{1/C} - 1\right) = K(S_1) - \frac{C}{C-1}\,\beta\, E_1$

Check infinite C limit!

Infinite heat capacity limit

- $P_i \rightarrow \frac{1}{Z}\, e^{-E_i/T_{fit}}$ with
- $\boldsymbol{T_{fit} = \frac{1}{\beta} = T \lim_{C\rightarrow\infty} e^{-S/C}}$

Finite subsystem corrections to infinite heat capacity limit

- $$T_1 = T \quad \frac{1}{1+1\cdot\frac{E_1}{CT}+\cdots}$$ traditional S-expansion

- $$T_1 = Te^{-S/C} \quad \frac{e^{S(E_1)/C}}{1+0\cdot\frac{E_1}{CT}+\alpha\cdot\frac{E_1^2}{C^2T^2}+\cdots}$$ **Our expression**

Traditional: T1 < T, falling in E1; Ours: T1 < T, but rising in E1 !

Gaussian approximation

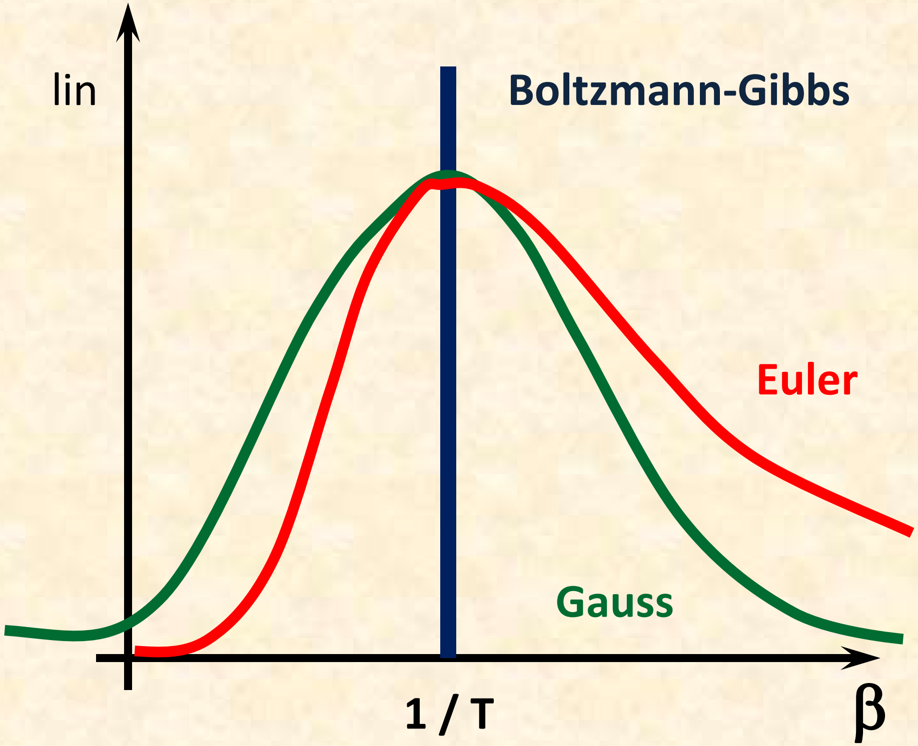
- Deviations from S=max equilibrium are traditionally considered as Gaussian:
- $\mathrm{P}(\Delta E) = e^{S(E_1)+S(E-E_1-\Delta E)} \approx$
$e^{-S'(E-E_1)\,\Delta E+\frac{1}{2}S''(E-E_1)\,\Delta E^2} \approx$
$\propto e^{-\frac{1}{T}\Delta E-\frac{1}{2CT^2}\Delta E^2}$

Gaussian approximation

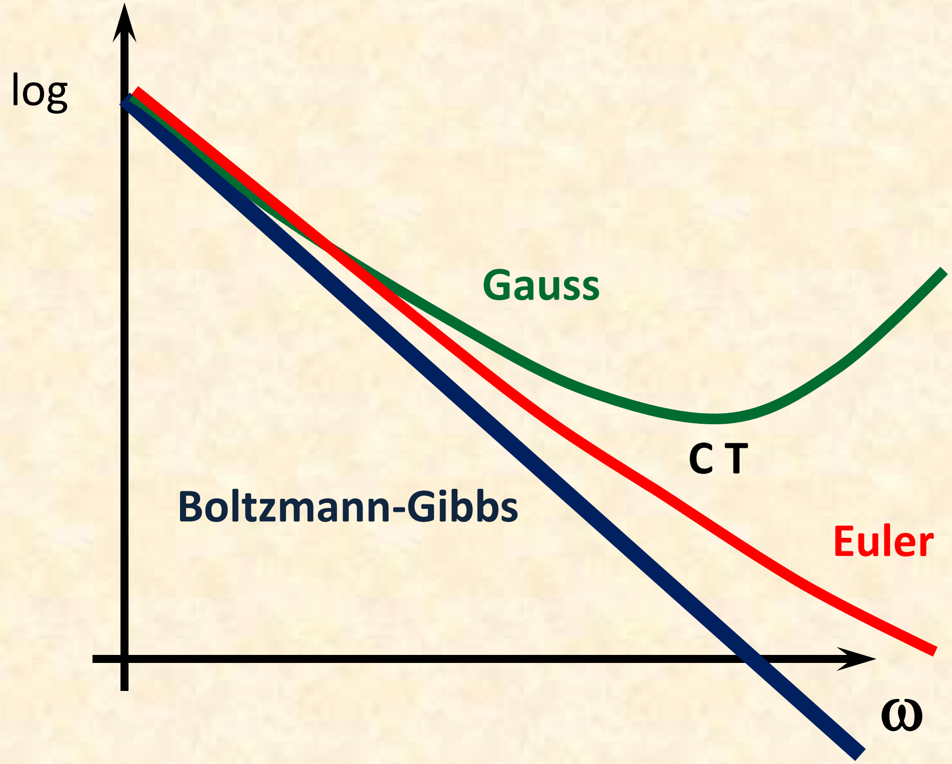
- After Legendre transformation also β fluctuates as Gaussian:
- $\mathrm{P}(\Delta\beta) \propto e^{-\frac{CT^2}{2}\Delta\beta^2 + \cdots}$
- Thermodynamic ”uncertainty” minimal

Gaussian approximation and beyond

Beta fluctuation



Particle spectra : $\langle e^{-\beta\omega} \rangle$



Application and Conclusion

- **Application:**
  - Reservoir = QGP at constant volume
  - Reservoir = QGP at constant pressure
  - Reservoir = QGP at constant entropy
  - Reservoir = classical Yang-Mills on lattice
  - Reservoir = (Schwarzschild) black hole
- **Conclusion:**
  - Why Tsallis / Rényi entropy ?
  - What is q ?
  - What is T ?

Heat capacity of QGP reservoir

- MIT bag model:
  - $E = V(\sigma T^4 + B), \quad p = \frac{\sigma T^4}{3} - B, \quad S = 4\sigma V T^3/3$

$$C = \frac{dE}{dT} = 4\sigma V T^3 + (\sigma T^4 + B)\frac{dV}{dT}$$

Heat capacity of QGP reservoir

- MIT bag model:
  - $E = V(\sigma T^4 + B), \quad p = \frac{\sigma T^4}{3} - B, \quad S = 4\sigma V T^3/3$

$$C_V = 4\sigma V T^3 = 3S, \qquad C_p = \infty, \qquad C_S = \frac{3}{4} S \left(1 - \frac{T^4}{T_0^4}\right)$$

V const. $\quad T_{fit} = T \lim_{C \to \infty} e^{-S/C} = T e^{-1/3} \approx 0.7\, T$

P const. $\quad T_{fit} = T \lim_{C \to \infty} e^{-S/C} = T$

S const. $\quad T_{fit} = T \lim_{C \to \infty} e^{-S/C} < T e^{-4/3} \approx 0.25\, T$

Heat capacity of QGP reservoir

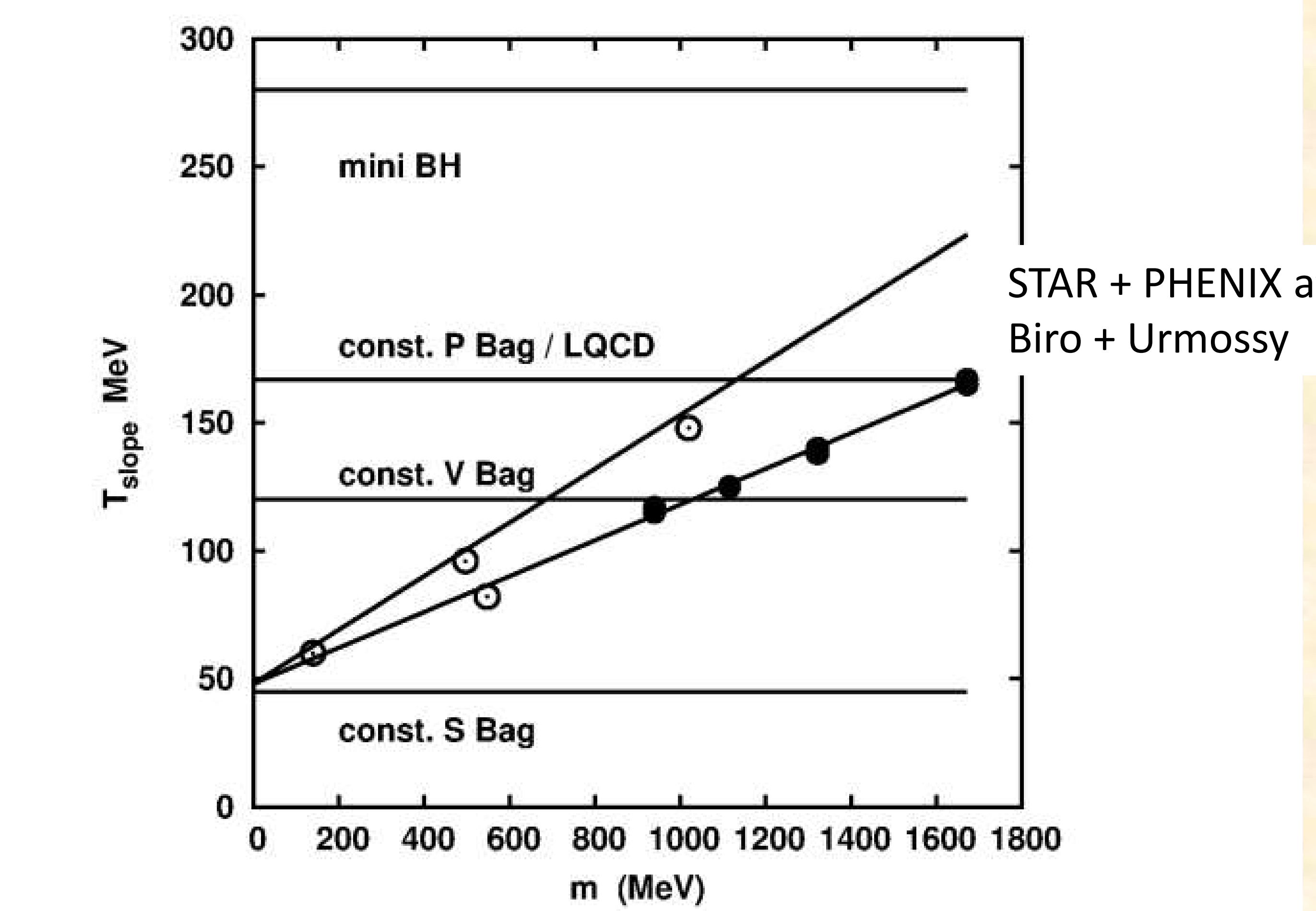
- Chaotic classical Yang-Mills:
  - $S(E) = C_0 \ln(1 + E/C_0T_0)$, constant heat capacity C !

$$T_{fit} = T_0, \qquad T = T_0 + E/C_0$$

- Schwarzschild black hole:
  - $S = \alpha E^2, \quad \frac{1}{T} = 2\alpha E, \quad C = -2\alpha E^2 = -2\,S$

$$T_{fit} = T \lim_{C\to\infty} e^{-S/C} = Te^{1/2} \approx 1.65\,T$$

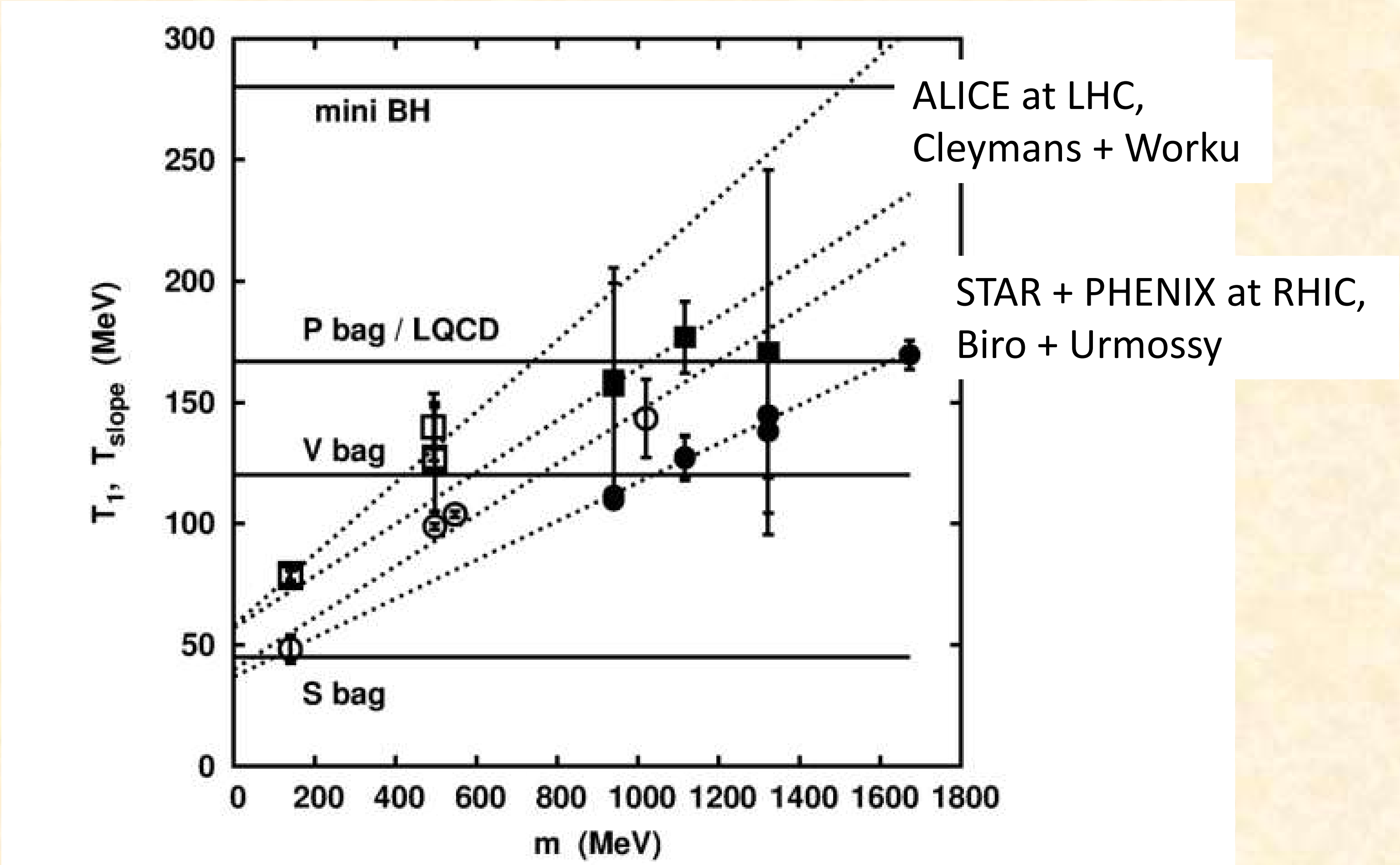
Fitted slopes



STAR + PHENIX at RHIC,
Biro + Urmossy

T1 = 1/beta = T exp(-S/C); Tslope = gothic T, fitted to experimental analysis

Fitted slopes

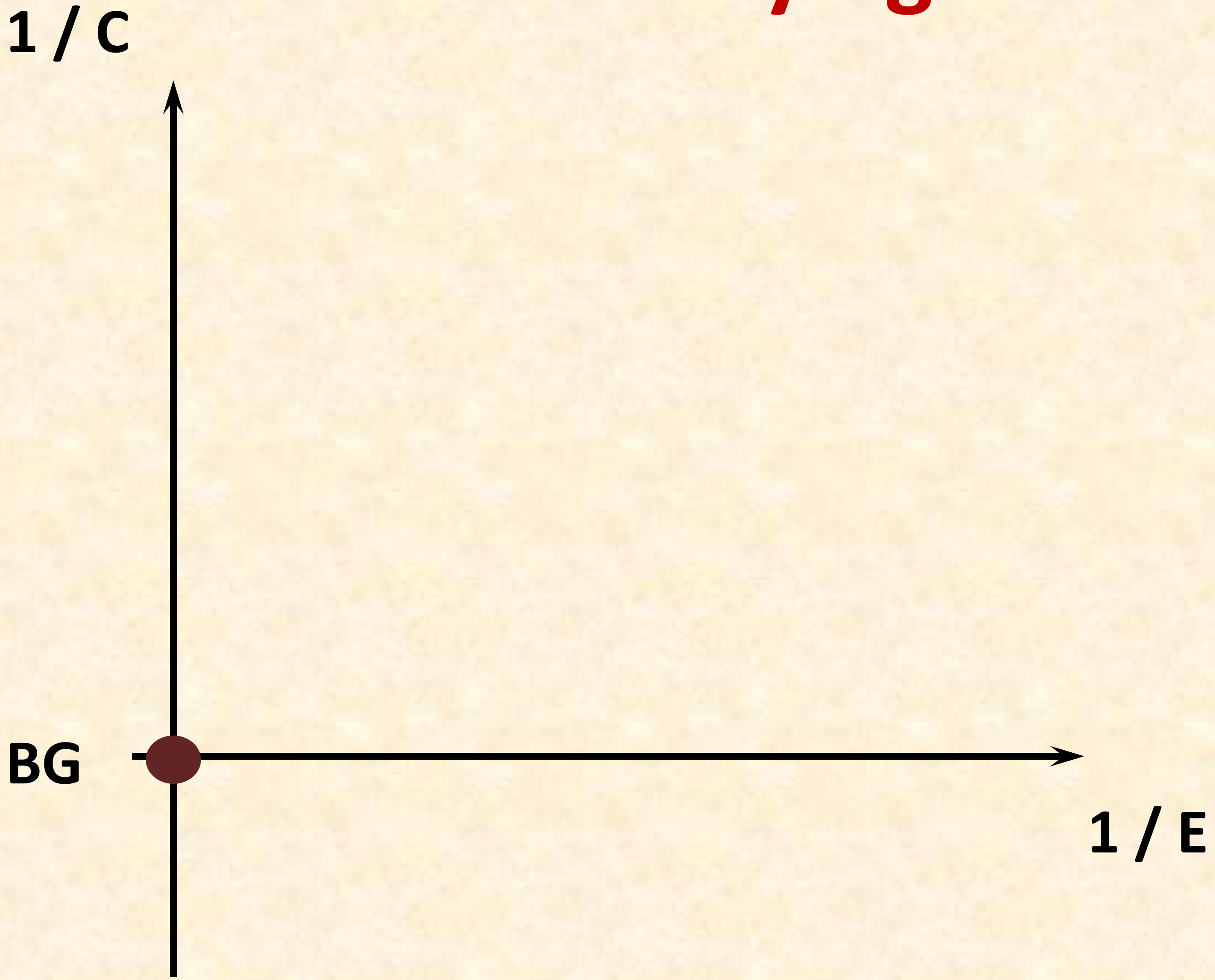


T1 = 1/beta = T exp(-S/C); Tslope = gothic T, fitted to experimental analysis

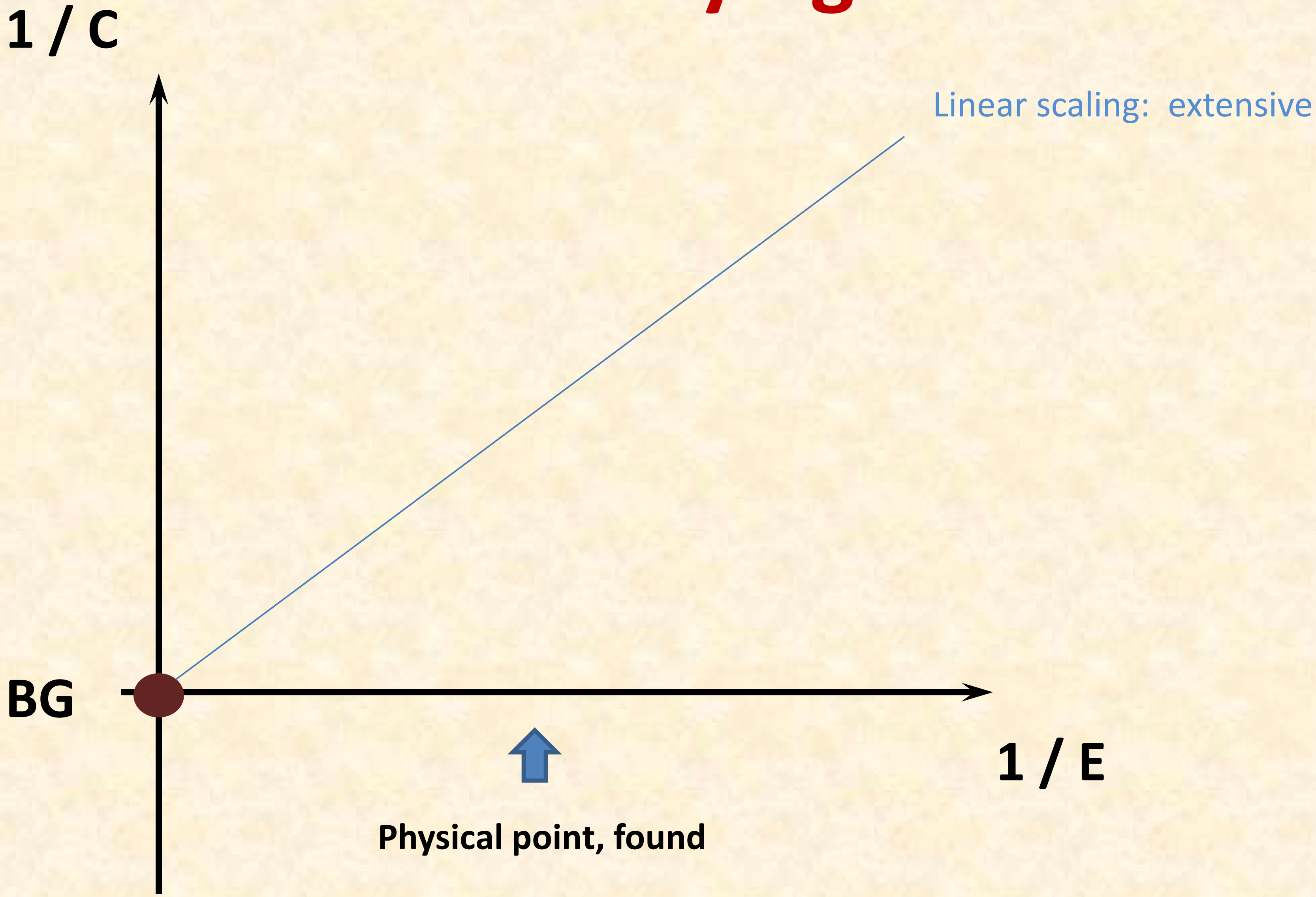
Universal Thermostat Independence

- ***Improved Canonical Approach***: **assumes statistical entanglement by using K(S) optimized to** $d^2 K(S(E)) / dE^2 = 0$, **rendering finite size corrections to one order higher**
- **Universally treats *finite heat capacity reservoirs* (but includes the infinite ones) by fixing** $K''(S) / K'(S) = a = -S''(E)/S'(E)^2$
- **Tsallis entropy is L(S), Rényi entropy is S**
- **Fitted Boltzmann-Gibbs temperature may differ from that of the finite reservoir in this case: it explains lower T fits by cut power-law**
- **For QGP with T = 175 MeV one has a**
  - **i) V=const fit T = 125 MeV,**
  - **ii) S=const QGP around 50,**
  - **iii) Yang-Mills fit T independent,**
  - **iv) mini BH fit T = 288 MeV too large**

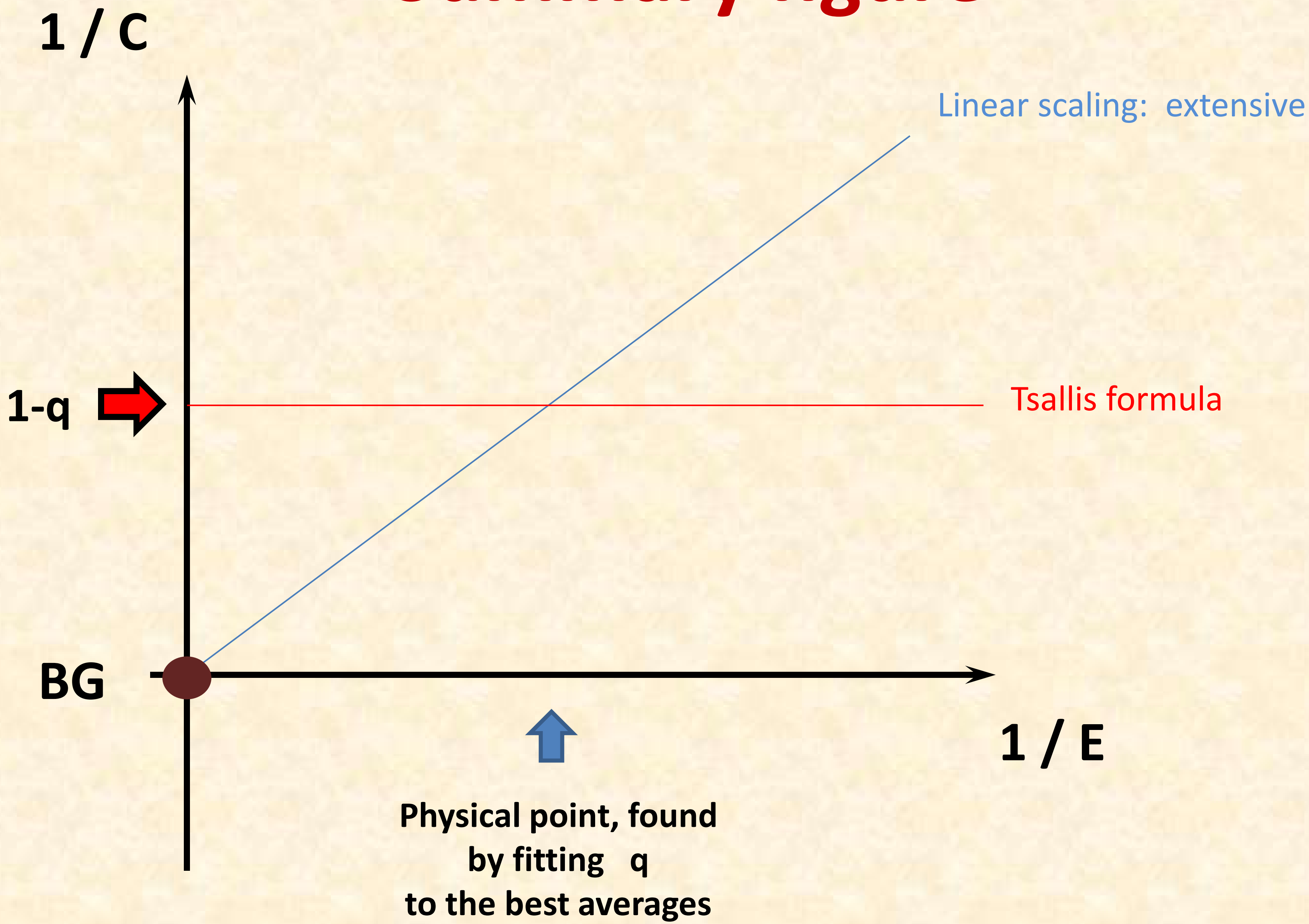
Summary figure



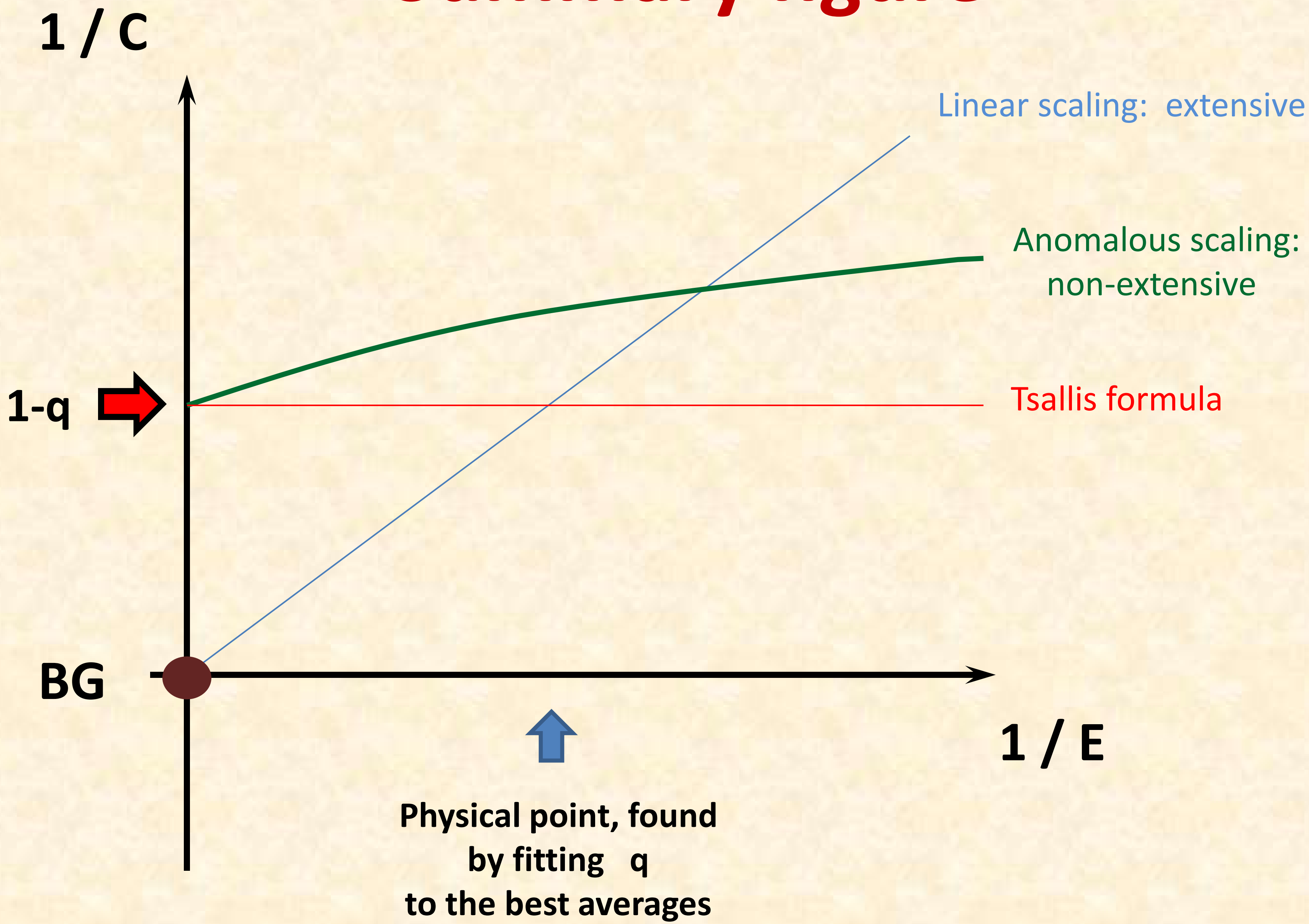
Summary figure



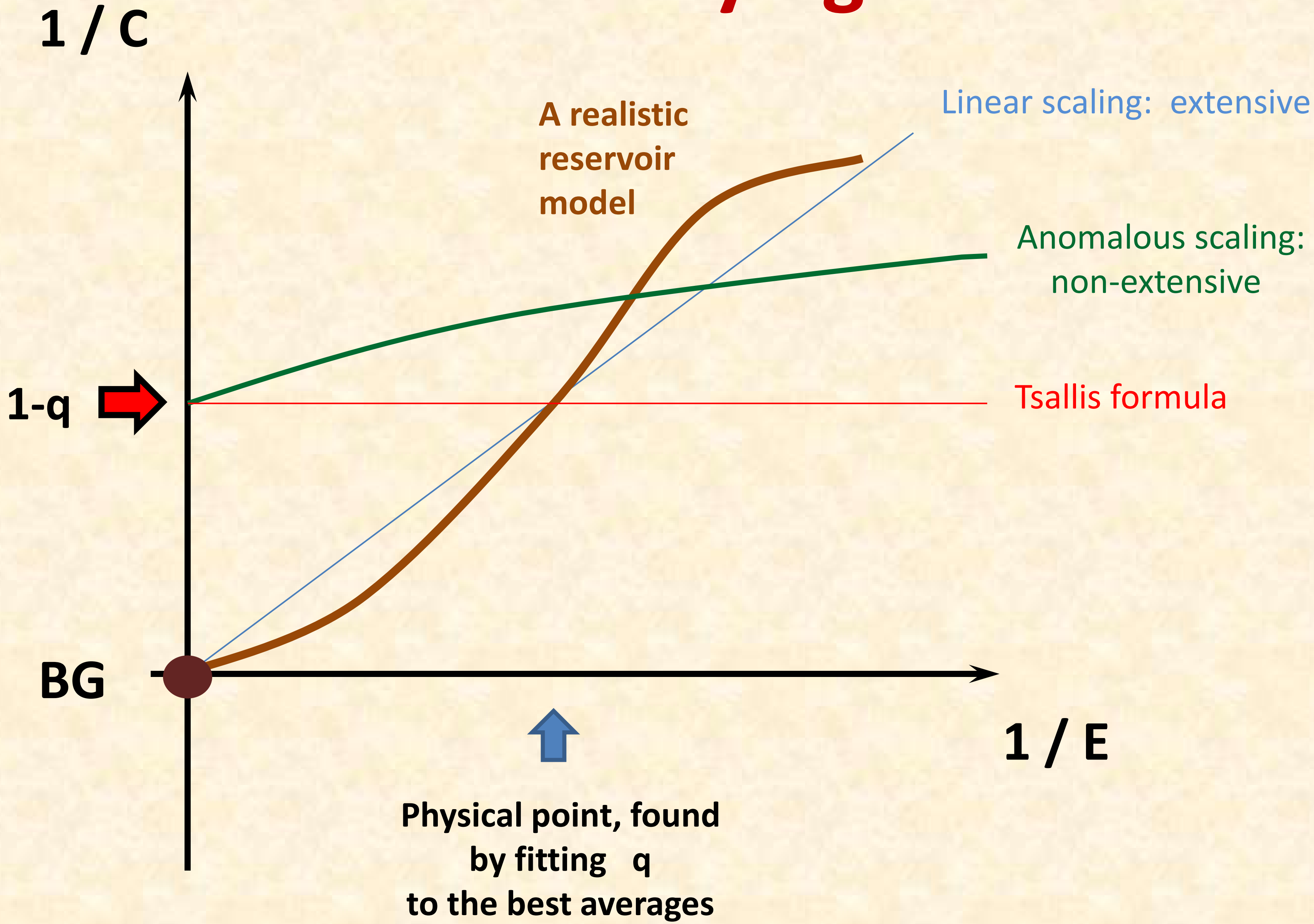
Summary figure



Summary figure



Summary figure



Summary figure